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ON STABILITY IN ONE CRITICAL CASE

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The problem on the stability of the trivial solution of an autonomous system of ordinary differential equations is solved in the critical case of one zero root, m pairs of pure imaginary roots, and q roots with negative real parts. It is proved that the presence of the zero root, as a rule, leads to instability, which can be detected already from the form of the second-order series expansion of the right hand sides of the equations. In the degenerate case necessary and sufficient stability conditions have been indicated for a model (simplified) system; it is shown that the absence of additional degeneracy the instability of the original system follows from that of the model. Sufficient conditions for the asymptotic stability and instability of the original system have been obtained under the fulfilment of the necessary stability conditions for the model system.

1. Preliminary remarks. We consider the system of ordinary differential equations

$$x_*^{\cdot} = Ax_* + X_*(x_*), \quad X_*(0) = 0 \tag{1.1}$$
$$x_* = (x_1^*, \ldots, x_n^*), \quad X_* = (X_1^*, \ldots, X_n^*)$$

where x_* and X_* are n-dimensional vectors of the Euclidean space E_n, $X_*(x_*)$ are holomorphic functions of x_*, $A = \| a_{rs} \|$ is a constant $n \times n$-matrix. The characteristic equation $\| a_{rs} - \delta_{rs}\lambda \| = 0$ has one zero root, m pairs of pure imaginary roots $\pm\lambda_i$ $(i = 1, \ldots, m)$, and q roots with negative real parts $(2m + q + 1 = n)$; if among the pure imaginary roots there are multiple ones, then simple elementary divisors correspond to them. By a nonsingular linear transformation we reduce system (1.1) to the form

$$y_*^{\cdot} = Q_* y_* + Y_*(y_*, z_*), \quad z_*^{\cdot} = P_* z_* + Z_*(y_*, z_*) \tag{1.2}$$
$$y_* = (y_1^*, \ldots, y_{2m+1}^*), \quad Y_* = (Y_1^*, \ldots, Y_{2m+1}^*),$$
$$z_* = (z_1^*, \ldots, z_q^*), \quad Z_* = (Z_1^*, \ldots, Z_q^*)$$

where the constant matrices Q_* and P_* have eigenvalues with zero and negative real parts, respectively.

It is well known [1,2] that the polynomial transformation

$$u_* = \sum_{l=1}^{l_*} u_*^{(l)}(y_*), \quad u_* = (u_1^*, \ldots, u_q^*), \quad u_*^{(l)} = (u_{*1}^{(l)}, \ldots, u_{*q}^{(l)})$$

where $u_{*i}^{(l)}$ $(i = 1, \ldots, q)$ are l-th-order forms in $y_1^*, \ldots, y_{2m+1}^*$, helps to reduce the stability problem for the trivial solution of (1.2) to solving it for a "shortened" system (a group of Eqs. (1.2) in y_*, in which z_* is replaced by $u_*(y_*)$), if the question can be resolved for the latter problem by forms of order up to l_*, inclusive, in the series expansion of functions $Y_*(y_*, u_*(y_*))$ in powers of y_*. It is assumed below that the transformation mentioned has been implemented and that $l_* \geqslant 2$. We write down the shortened system

$$\begin{aligned}
&\zeta^{\cdot} = F(\zeta, y, \bar{y}) \qquad (1.3)\\
&\dot{y} = \Lambda y + Y(\zeta, y, \bar{y}), \quad \bar{y}^{\cdot} = -\Lambda y + \bar{Y}(\zeta, y, \bar{y})\\
&y = (y_1, \ldots, y_m), \quad Y = (Y_1, \ldots, Y_m), \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)
\end{aligned}$$

Here ζ is a real variable, y and $\bar{y}$ are complex-conjugate vectors, Λ is the diagonal matrix of the pure imaginary eigenvalues, and the expansions of function F and of the complex-conjugate vector-valued functions Y and $\bar{Y}$ in series with respect to ζ, y and $\bar{y}$ start with second-order terms. We transform system (1.3) by the nonlinear replacements [3]

$$\begin{aligned}
&\zeta = x + \sum_{l=2}^{l_*} \Xi^{(l)}(x, u, v)\\
&y = u + \sum_{l=2}^{l_*} \Phi^{(l)}(x, u, v), \quad \bar{y} = \sum_{l=2}^{l_*} \Psi^{(l)}(x, u, v)\\
&\Phi^{(l)} = (\Phi_1^{(l)}, \ldots, \Phi_m^{(l)}), \quad \Psi^{(l)} = (\Psi_1^{(l)}, \ldots, \Psi_m^{(l)})
\end{aligned}$$

to normal form up to terms of order l_*, inclusive, where x is a real variable, u and v are complex-conjugate vectors with components u_i and v_i $(i = 1, \ldots, m)$, and the complex-conjugate functions $\Phi_i^{(l)}$ and $\Psi_i^{(l)}$ are l th-order forms in x, u, v. Then in the new variables we obtain the following system [1, 2]:

$$\begin{aligned}
&x^{\cdot} = \sum_{l=2}^{l_*} X^{(l)}(x, u, v) + X(x, u, v) \qquad (1.4)\\
&u^{\cdot} = \Lambda u + \sum_{l=2}^{l_*} U^{(l)}(x, u, v) + U(x, u, v)\\
&v^{\cdot} = -\Lambda v + \sum_{l=2}^{l_*} V^{(l)}(x, u, v) + V(x, u, v)
\end{aligned}$$

The expansions of function X and of the complex-conjugate vector-valued functions U and V start with terms of order higher than l_*, while $X^{(l)}$ is a real and $U^{(l)}$ and $V^{(l)}$ are complex-conjugate vector-valued forms of order l, such that

$$\begin{aligned}
&X^{(l)} = \sum_{p_0+|k_0|+|l_0|=l} R_{p_0 k_0 l_0} x^{p_0} u_1^{k_{01}} \ldots u_m^{k_{0m}} v_1^{l_{01}} \ldots v_m^{l_{0m}}\\
&U_s^{(l)} = \sum_{p_s+|k_s|+|l_s|=l} R_{p_s k_s l_s} x^{p_s} u_1^{k_{s1}} \ldots u_m^{k_{sm}} v_1^{l_{s1}} \ldots v_m^{l_{sm}} \quad (s = 1, \ldots, m)
\end{aligned}$$

The only nonzero coefficients $R_{p_0 k_0 l_0}$ and $R_{p_s k_s l_s}$ are those for which the integer-numerical vectors

$$\begin{aligned}
&k_s = (k_{s1}, \ldots, k_{sm}), \quad l_s = (l_{s1}, \ldots, l_{sm}), \quad k_{sj}, l_{sj} \geqslant 0\\
&(s = 0, 1, \ldots, m)
\end{aligned}$$

satisfy one of the relations [3]

$$\langle(k_0 - l_0),\ \Lambda\rangle = 0,\quad p_0 + |k_0| + |l_0| = l \tag{1.5}$$
$$\langle(k_s - l_s),\ \Lambda\rangle = \lambda_s,\ p_s + |k_s| + |l_s| = l \quad (s = 1, \ldots, m)$$
$$p_0, p_s \geqslant 0,\quad |k_s| = \sum_{j=1}^{m} k_{sj},\quad |l_s| = \sum_{j=1}^{m} l_{sj}$$

where p_0 and p_s are integers. We can satisfy ourselves that relations (1.5) are fulfilled identically with respect to λ_s if

$$k_{0j} = l_{0j},\quad k_{sj} = l_{sj} + \delta_{sj} \quad (s, j = 1, \ldots, m)$$

where δ_{sj} is the Kronecker symbol. If l is even, then $p_0 = 0, 2, \ldots, l$; $p_s = 1, 3, \ldots, l - 1$; if l is odd, then $p_0 = 1, 3, \ldots, l$; $p_s = 0, 2, \ldots, l - 1$. But if Λ satisfies the internal resonance condition [4]

$$\langle P_r,\ \Lambda\rangle = 0,\quad P_r = (P_{r1}, \ldots, P_{rm}),\ |P_r| = K\ (r = 1, \ldots, r_*) \tag{1.6}$$

then in Eqs. (1.4) for $l = K - 1, \ldots, l_*$ there appear additional internal resonance terms supplied by (1.5) and (1.6). In the presence of multiple roots $(K = 2)$ the additional internal resonance terms appear for $l = 2, \ldots, l_*$.

2. Instability theorem. We pass to polar coordinates by the formulas $u_s = \rho_s \exp(i\theta_s)$ and $v_s = \rho_s \exp(-i\theta_s)$ $(s = 1, \ldots, m)$. Then, by what was said above, the group of equations for x and ρ_s has the form

$$x^{\cdot} = gx^2 + \sum_{i=1}^{m} a_i\rho_i^2 + X_0(\rho, \theta) + X_1(x, \rho, \theta) \tag{2.1}$$
$$\rho_s^{\cdot} = b_s x\rho_s + R_{0s}(x, \rho, \theta) + R_{1s}(x, \rho, \theta) \quad (s = 1, \ldots, m)$$

Here X and R_{1s} are holomorphic functions of x and ρ_s, with coefficients that are polynomials in $\cos\theta_s$ and $\sin\theta_s$ and containing terms of order no lower than third relative to x and ρ_s; X_0 and R_{0s} are the resonance terms supplied by (1.5) and (1.6) when $K = 2, 3$ (and equal identically to zero in the absence of multiple roots and of third-order resonance), being second-degree polynomials in ρ_s and in x and ρ_s, respectively, with coefficients linear in $\cos\theta_s$ and $\sin\theta_s$; g, a_i and b_s are real constants.

We consider the functions

$$V = x,\quad W = \sum_{s=1}^{m} \rho_s^2 - x^{2(1+\gamma)}$$

where γ is a positive constant subject to choice. The derivative of W relative to system (2.1) is

$$\frac{1}{2} W^{\cdot} = x \sum_{s=1}^{m} b_s\rho_s^2 - (1 + \gamma)\, x^{1+2\gamma} \Big(gx^2 + \sum_{i=1}^{m} a_i\rho_i^2\Big) +$$
$$\sum_{s=1}^{m} \rho_s (R_{0s} + R_{1s}) - (1 + \gamma)\, x^{1+2\gamma} (X_0 + X_1)$$

In a neighborhood of zero $x^2+\rho_1^2+\ldots+\rho_m^2<A,\ A>0$, we consider the domain $W\leqslant 0$ which belongs to domain $VV^{\cdot}>0$ for a sufficiently large γ. We determine the value of derivative $W^{\cdot}$ on the boundary $W=0$

$$\frac{1}{2}W_0^{\cdot}=-(1+\gamma)gx^{3+2\gamma}+x\sum_{s=1}^{m}b_s\rho_s^2+\sum_{s=1}^{m}\rho_sR_{0s}+o(x^{3+2\gamma})$$

Let $g\gtrless 0$. By choosing γ sufficiently large we can achieve $W_0^{\cdot}\lessgtr 0$. The functions V and W satisfy Chetaev's two-function instability theorem [5]. We state the following result.

Theorem 1. If $g\neq 0$, the trivial solution of system (2.1) (and hence, of (1.1)) is Liapunov-unstable.

Thus, the presence of one zero root in the linear-approximation characteristic equation, when the others are pure imaginary and with negative real parts, leads, as a rule, to instability which can now be detected by second-order forms. Obviously, the case $g=0$ should be considered degenerate. We note that to find g it is enough to take the linear-approximation system to the canonic form and to pick out, in the equation for the variable corresponding to the zero root, the coefficient of the square of this variable. The coefficient picked out is g.

3. Case $g=0$. Henceforth we assume the absence of multiple roots. At first let third-order resonances not be present in the system. Then in system (2.1) $X_0=R_{0s}\equiv 0\ (s=1,\ldots,m)$ and we have

$$x^{\cdot}=\sum_{i=1}^{m}a_i\rho_i^2+X_1(x,\rho,\theta)$$
$$\rho_s^{\cdot}=b_sx\rho_s+R_{1s}(x,\rho,\theta)\quad(s=1,\ldots,m) \tag{3.1}$$

Let us show that the existence of pairs of coefficients a_{s*} and b_{s*} such that $a_{s*}b_{s*}>0$ leads to instability. Indeed, in this case, for a model system truncated up to cubic terms there exists a ray-type growing solution

$$\rho^{\cdot}=\sqrt{a_{s*}b_{s*}}\rho^2,\quad x=k\rho,\quad \rho_{s*}=\rho,\quad \rho_i=0$$
$$(i=1,\ldots,m;\ i\neq s_*),\quad k^2=\frac{a_{s*}}{b_{s*}}$$

The instability of the complete system is proved in the usual manner by the scheme in [4].

Now suppose that $a_ib_i<0\ (i=1,\ldots,m)$. Then a sign-definite integral

$$x^2-\sum_{i=1}^{m}\frac{a_i}{b_i}\rho_i^2=\text{const}$$

obtains, whose existence proves the stability of the model system. If one of the coefficients a_i and b_i is zero, while the rest are such that $a_jb_j<0\ (j=1,\ldots,m;\ j\neq i)$, then the model system has the growing solution

$$x=x_0,\ \rho_i=\rho_{i0}e^{b_ix_0t},\ \rho_j=0\ (j=1,\ldots,m;\ \ j\neq i),\ a_i=0,\ b_i\neq 0$$

$$x = x_0 + a_i \rho_{i0}^2 t, \quad \rho_i = \rho_{i0}, \quad \rho_j = 0 \ (j = 1, \ldots, m; \ j \neq i),$$
$$a_i \neq 0, \ b_i = 0$$

where x_0 and ρ_{i0} are constants. However, we have been unable to prove the instability of the complete system in these cases.

Theorem 2. A necessary and sufficient stability condition for a model system truncated up to cubic terms is $a_i b_i \leqslant 0 \quad (i = 1, \ldots, m)$, and equality is achieved only if $a_i = b_i = 0$. If a pair of coefficients a_{s*} and b_{s*} exists such that $a_{s*} b_{s*} > 0$, then the trivial solution of system (3.1) (and, hence, of (1.1)) is Liapunov-unstable.

Now let a third-order resonance, say $\lambda_1 = 2\lambda_2$, hold in the system. Restricting ourselves (without loss of generality) to the case $m = 2$, let us show that the addition of a weak resonance [6] to a neutral zero root (all $a_i b_i < 0$) can lead to the instability of the whole system. According to the necessary and sufficient conditions for the weakness of the resonance $\lambda_1 = 2\lambda_2$ [7], system (2.1), after the addition of equations in θ_s under the assumptions made, can be written as

$$\begin{aligned} x^{\cdot} &= a_1 \rho_1^2 + a_2 \rho_2^2 + X_1 (x, \rho_1, \rho_2, \theta_1, \theta_2) \\ \rho_1^{\cdot} &= b_1 x \rho_1 + c \rho_2^2 \cos \theta + R_{11} (x, \rho_1, \rho_2, \theta_1, \theta_2) \\ \rho_2^{\cdot} &= b_2 x \rho_2 + \rho_1 \rho_2 \cos \theta + R_{12} (x, \rho_1, \rho_2, \theta_1, \theta_2) \\ \rho_1 \rho_2 \theta^{\cdot} &= -(c \rho_2^2 + \rho_1^2) \rho_2 \sin \theta + \Theta (x, \rho_1, \rho_2, \vartheta_1, \vartheta_2) \\ \theta &= 2\theta_2 - \theta_1, \ a_1 b_1 < 0, \quad a_2 b_2 < 0, \ c < 0 \end{aligned} \tag{3.2}$$

where the series expansions of function Θ in x, ρ_1 and ρ_2, with coefficients that are polynomials in $\sin \theta_s$ and $\cos \theta_s \ (s = 1, 2)$, start with terms of order no lower than fourth. Let us consider the model system obtained from (3.2) by discarding X_1, R_{11}, R_{12} and Θ. Then, if γ, γ_1, and $\gamma_2 \ (\gamma_{1,2} > 0)$ exist such that the conditions

$$\frac{a_1 \gamma_1^2 + a_2 \gamma_2^2}{\gamma} = b_1 \gamma + \frac{c \gamma_2^2}{\gamma_1} = b_2 \gamma + \gamma_1$$

are fulfilled, which is possible under specific relations between a_s, b_s and c, then the model system has a ray-type growing solution

$$x = \gamma \rho, \ \rho_1 = \gamma_1 \rho, \ \rho_2 = \gamma_2 \rho, \ \rho^{\cdot} = b \rho^2, \quad \theta = 0, \ b > 0 \tag{3.3}$$

The instability of the complete system when ray (3.3) exists can be proved by the scheme in [4].

4. Investigation on higher-order terms. We now assume that the system does not have a Kth-order resonance, $2 \leqslant K \leqslant N + 1$ $(N \geqslant 3)$ and that the necessary conditions for stability with respect to second-order terms, i.e., $g = 0$ and $a_i b_i \leqslant 0 \ (i = 1, \ldots, m)$ and equality $a_i b_i = 0$ is possible only if $a_i = b_i = 0$, are fulfilled. In this case, obviously, we can always achieve $a_i = -b_i \ (i = 1, \ldots, m)$ by a change of variables. In (2.1) we pass to $(m + 1)$ - dimensional spherical coordinates by the formulas

$$x = r\cos\varphi_1, \quad \rho_s = r\cos\varphi_{s+1}\prod_{j=1}^{s}\sin\varphi_j, \quad \rho_m = r\prod_{j=1}^{m}\sin\varphi_j$$
$$(s = 1, \ldots, m-1)$$
$$0 \leqslant \varphi_1 \leqslant \pi, \quad 0 \leqslant \varphi_j \leqslant \frac{\pi}{2} \quad (j = 2, \ldots, m)$$

In the new variables we have

$$r^{\cdot} = r^2R^{(2)} + r^3R^{(3)} + \ldots + r^NR^{(N)} + \ldots \tag{4.1}$$
$$\varphi_1^{\cdot} = r\Phi_1^{(1)} + r^2\Phi_1^{(2)} + \ldots + r^{N-1}\Phi_1^{(N-1)} + \ldots, \left(\prod_{j=1}^{s}\sin\varphi_j\right)\varphi_s^{\cdot} =$$
$$r\Phi_s^{(1)} + \ldots + r^{N-1}\Phi_s^{(N-1)} + \ldots (s = 2, \ldots, m)$$
$$\Phi_1^{(1)} = \left(\sum_{i=1}^{m-1} b_i\cos^2\varphi_{i+1}\prod_{j=2}^{i}\sin^2\varphi_j + b_m\prod_{j=2}^{m}\sin^2\varphi_j\right)\sin\varphi_1$$

where $R^{(l)}$ and $\Phi_s^{(l-1)}$ $(s = 1, \ldots, m;\ l = 2, \ldots, N)$ are polynomials in $\sin\varphi_j$ and $\cos\varphi_j$ $(j = 1, \ldots, m)$ and the terms not written out are of the following orders relative to r: higher than N in the equation in r and higher than $N-1$ in the equations in φ_i; $R^{(2)} \equiv 0$.

We consider the following angle values:

$$\varphi_1 = \varphi_1^{\circ} = 0, \pi; \quad \varphi_j = \varphi_j^{\circ} = \text{const} \quad (j = 2, \ldots, m) \tag{4.2}$$

The following statement is valid.

Theorem 3. If the condition

$$R^{(3)}(\varphi^{\circ}) = \ldots = R^{(N-1)}(\varphi^{\circ}) = 0, \; R^{(N)}(\varphi^{\circ}) > 0$$

is fulfilled on even one angle value (4.2), the trivial solution $r = 0$ is Liapunov-unstable. However, if

$$R^{(3)}(\varphi^{\circ}) = \ldots = R^{(N-1)}(\varphi^{\circ}) = 0, \; R^{(N)}(\varphi^{\circ}) < 0$$

on all values (4.2) and all coefficients b_s $(s = 1, \ldots, m)$ are of one sign, then asymptotic stability obtains.

Note. The case when among the coefficients b_s $(s = 1, \ldots, m)$ there is even one change of sign or there are zero coefficients, requires an individual analysis.

Proof. We assume that reduction to normal form (1.4) up to Nth-order terms, inclusive, i.e., $l_* \geqslant N$ in (1.4), has been carried out. Consequently, the functions $\Phi_1^{(l)}$ contain $\sin\varphi_1$ as factors, while the functions $\Phi_s^{(l)}$ $(s = 2, \ldots, m)$ contain

$$\cos\varphi_s\sin\varphi_1\prod_{j=1}^{s}\sin\varphi_j$$

i.e.,

$$\Phi_1^{(l)}|_{(4\cdot 2)} = 0, \quad \frac{1}{\sin\varphi_1}\Phi_s^{(l)}|_{(4\cdot 2)} = 0 \quad (s = 2, \ldots, m;\ l = 2, \ldots, N-1)$$

Therefore, in the first case a ray-type solution

$$r^{\cdot} = r^N R^{(N)}(\varphi^\circ), \quad R^{(N)}(\varphi^\circ) > 0 \tag{4.3}$$

where φ_i° $(i = 1, \ldots, m)$ are from (4.2), exists for the system truncated up to $(N+1)$ st-order terms. The instability of the complete system is proved by constructing a Chetaev function in a neighborhood of ray (4.3), as in [4].

To prove the theorem's second assertion we consider the function

$$V = r\exp(h\cos\varphi_1), \quad h = \text{const} \tag{4.4}$$

The derivative of function (4.4) relative to Eqs. (4.1) is

$$V^{\cdot} = rV\Big\{rR^{(3)} + \ldots + r^{N-2}R^{(N)} - h\Big[\sum_{i=1}^{m-1} b_i\cos^2\varphi_{i+1}\prod_{j=2}^{i}\sin^2\varphi_j +$$
$$b_m\prod_{j=2}^{m}\sin^2\varphi_j\Big]\sin^2\varphi_1 - h\sin\varphi_1[r\Phi_1^{(2)} + \ldots + r^{N-2}\Phi_1^{(N-1)}] + \ldots\Big\}$$

where the terms not written out are of order higher than $N-2$ relative to r. Since the functions $R^{(l)}$ and $\Phi_1^{(l-1)}$ $(l = 3, \ldots, N)$ (except $R^{(N)}$) contain $\sin^2\varphi_1$ and $\sin\varphi_1$, respectively, as factors, while all b_i are of one sign, say, positive (which can always be achieved by replacing x by $-x$ in (2.1)), we can, by choosing a sufficiently large $h > 0$, achieve the negative definiteness of $V^{\cdot}$ in the domain $r < A$, where A is some sufficiently small positive number. A function V thus defined satisfies Liapunov's asymptotic stability theorem [8].

The theorem proved yields the following simple criterion. If under the assumptions in Sect.4 we have

$$x^{\cdot} = A_0x^N + A_1x^{N+1} + \ldots$$

in (1.4) when $u = v = 0$, then the trivial solution is unstable when N is even and when $A_0 > 0$ if N is odd. However, if N is odd and $A_0 < 0$, then the trivial solution is asymptotically stable if there are no changes of sign among the b_s $(s = 1, \ldots, m)$.

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